NOTES ON THE ELECTRIC MOTOR AS A PHYSICAL SYSTEM

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Classification

PACS : 05.45.-a 41.20.-q 84.60.-h

Keywords : Electric motor Maxwellian bicylinder model Nonlinear dynamical formalism

Abstract

The aim of this work is to present the electric motor as a model system in nonlinear dynamics. To this end, I derive the motor's equations of motion starting from classical electromagnetism, and give two examples of results which can be obtained by solving these equations using formal nonlinear dynamical procedures. The first of these examples deals with synchronization in coupled permanent magnet motors, while the second shows stability of an induction motor driven by an electronic commutator.

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Introduction

"Figaro la, Figaro qua, Figaro su, Figaro giu ...... Sono il factotum della citta."

– The Barber of Seville

The electric motor is without doubt the most ubiquitous machine in today's technologically advanced world. From the milliwatt level motors in hard disks and cd drives to the ten-thousand horsepower giants which propel ships; from the ceiling fan where you just close a circuit and the thing runs to the locomotive where a computer changes the voltage several thousand times per second to achieve the best acceleration, electric motors are the factotum of modern industry and technology. As per one author's estimate [1], nearly two-thirds of all the electric power generated in the United States of America finds its way into an electric motor. The original inventor of the motor (a dc motor) was none other than Michael Faraday, one of the founders of classical electromagnetism. The ac motor (which has wider applicability today than the dc motor) was invented by another great physicist, Nikola Tesla, after whom the unit of magnetic field is named. And yet, the techniques in use today for analysing this machine are quite far removed from the original physical approach visualized by the pioneers.

The subfield in physics called nonlinear dynamics and chaos theory is tailormade to analyse the behaviour of natural as well as artificial dynamical systems. Examples of the latter are generally taken from engineering, for example the Kapitsa pendulum (model of vibrating machinery), the van der Pol oscillator (model of an electric circuit) and the neural network (artificial imitation of a brain). The simple circuit [2] consisting of an inductor, a resistor, two capacitors and a negative-resistivity diode, invented by the electrical engineer Leon Chua, did not take long to bridge the gap and become a staple of the physicist's diet. But a physics-based analysis of electric motors is today a rarity.

Yet it is worth noting that the basic principles behind the operation of motors are still described in electromagnetic terms – Ampere's law to explain how currents produce magnetic fields, Lenz's law to account for induced emf and $\vec{F} = i\vec{l}\times\vec{B}$ (where i is the current) to demonstrate why the rotor develops a torque. What is absent is the quantitative development of these fundamental laws – when plots and numbers are the requirement, the approach changes from the electromagnetic one to the equivalent circuit model. The advantage of this model is that it enables a very simple description of the steady state operating characteristics. When it comes to dynamic modeling however, the theory, which now assumes a two-reaction form, is much more complex and because of this, in the words of Krishnan, "[the dynamic model] is considered difficult and is avoided by many practising engineers".

In the first part of this work, my aim is to construct the dynamic motor model using the physically transparent electromagnetic principles as the starting point. We will see that the equations of motion are obtained in a relatively few number of logically connected steps. In the second part, I will attack the dynamical equations using the standard tools of nonlinear dynamics and chaos theory and use this formalism to derive two results. Since I am writing this article for a wide audience, I will include a largish amount of introductory material, and readers who are familiar with it can skip the respective passages. The outline of this paper is as follows. In Section 1 I will briefly review the motoring principle and then give a detailed account of the modeling of a general motor. In Section 2 I will demonstrate synchronization in an array of coupled permanent magnet motors and stabilizing dynamics of an induction motor driven by an electronic commutator. I will then finish the paper with a Conclusion.

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1. Modeling the electric motor

"When you read you begin with A-B-C, when you sing you begin with do-re-mi ..... when I write I begin with $\vec{i}\times\vec{B}$."

– Adapted from The Sound of Music

The aim of this Section is to present a model of a general motor structure, starting from classical electromagnetic theory. Once the general equation is derived, the forms corresponding to various kinds of motors and drives can easily be derived. Because of the way the modelling will be carried out, it will hereafter be known as the Maxwellian bicylinder (MBC) model.

1.1 The Basic motoring principle – types of motors

I will start from the material covered in introductory physics books like the ones I mentioned in the Introduction. The basic schematic representation of a dc motor is shown in Figure 1aL. A current loop ABCD in the plane of the page is kept in an uniform magnetic field also in the plane of the page and parallel to AD and BC. In the figure, this field is shown as being generated by a permanent magnet though an electromagnet may also be used. Applying the rule $\vec{F} = i\vec{l} \times \vec{B}$ there is no force on arms AD and BC, there is an upward force on AB and a force of equal magnitude but opposite direction on CD. These two forces combine to form a couple which results in a (by definition positive) torque on the loop and causes it to rotate. Just this is not sufficient to make a motor because the torque direction will reverse as soon as the loop has turned through 90^o and so the loop will execute oscillations rather than rotations. If however the current in the loop can be reversed the moment the torque starts becoming negative, then the torque will again become positive and allow continuous rotational motion. This is just what is done by the device called the mechanical commutator shown in Fig. 1aR. As soon as the loop reaches the crossed position (where the torque on it is zero), the commutator causes the current in it to flip direction so that the torque never becomes negative. An easy bit of jargon at this point : the permanent magnets creating the field are static so they are called the *stator* while the current coil is rotating so it is called the *rotor*. Rotor is sometimes also called armature, a terminology which I will not use here. And of course since the motor is being powered by dc currents, it is called a dc motor (DCM).

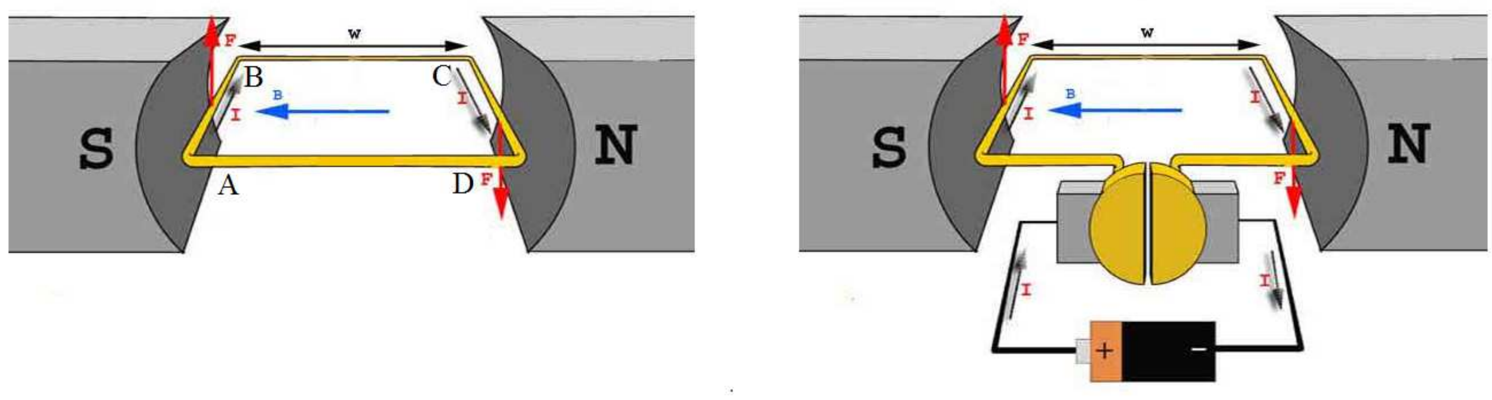


Figure 1a: *Schematic representation of a dc motor. This image is taken from www.electrical4u.com. 1aL shows a current loop in a magnetic field created by the two pole pieces. 1aR shows the commutator included in the circuit.*

Now suppose I interchange stator and rotor i.e. I bring the permanent magnet into the rotor and make the stator an electric circuit which produces a magnetic field. A torque will get generated on the rotor which will try to align with the stator's field (like a compass needle aligning with the earth's field). By analogy with the previous case, this setup should show a continuous positive torque output if I can flip the direction of the stator field (and hence the stator current) at the appropriate points in the rotation cycle. Of course, a mechanical commutator cannot be used here but I can always reverse the stator field electronically. To get smoother operation, I can try creating a continuously rotating magnetic field instead of just a flipping one. As it happens, there is a neat way of doing this, which I will discuss a little later. This motor is the permanent magnet synchronous motor (PMSM). Since there is a question of reversing the direction of stator current, this is an alternating current or ac motor. The simplicity of its operating principle belies the fact that this is the most advanced type of motor found in today's market.

To get the third type of motor I retain the stator of the PMSM but have as rotor the simple loop of Fig. 1aL. Now if this loop is kept in the rotating magnetic field, then there will be a continuous change of magnetic flux through it. By Faraday's law, emf will be induced in the loop. Lenz's law states that the effect of this change is to oppose the cause which produces it. The cause here is the rotation of the magnetic field relative to the loop; to oppose this cause, the loop too will start rotating in a bid to catch up with the stator field. This is the working principle of the squirrel cage induction motor (the standard abbreviation for this is just IM – the squirrel cage refers to a detail of the rotor construction and is assumed unless explicitly stated otherwise). Since this too requires a rotating stator field, it falls under the category of ac motor. IM is in fact the most widely used motor in industry today.

Thus we have already covered the basic principles behind the three fundamental types of motors. Our ultimate aim however is to get a quantitative and not just a qualitative picture. To this end we will take a closer look at the stators and rotors of actual motors. I will focus primarily on the two types of ac motor because their performance characteristics are far superior to those of dc motors.

1.2 The Structure of the stator and rotor; the motor drive

For reasons of symmetry, the stator and rotor are generally both in the form of cylinders and consist of electric wires or conducting bars embedded in a core made of a magnetic material like iron or steel. Fig. 1b shows a stator core with 24 slots hewn into it to place the windings. The vast majority of stators of ac motors are in fact three-phase, a term which refers both to the kind of voltage typically supplied to them as well as to the geometric layout of their windings. Three phase ac voltage refers to a combination of three waveforms $A\cos(\Omega t)$, $A\cos(\Omega t+2\pi/3)$ and $A\cos(\Omega t+4\pi/3)$. Of course, 'combination' cannot refer to just a linear superposition as the sum of the three waveforms is exactly zero.

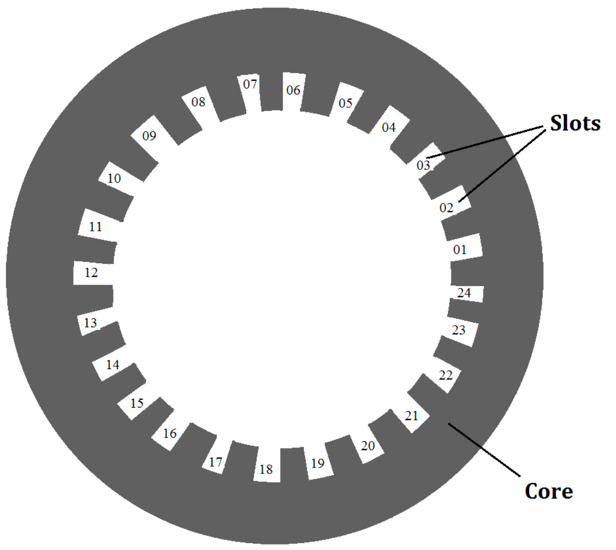


Figure 1b : *Schematic top view of the stator core of a three phase motor. It is made of a magnetic material and 24 labelled slots have been hewn into it to place the windings. More realistic images may be found in the classic book by Stephen Chapman [7].*

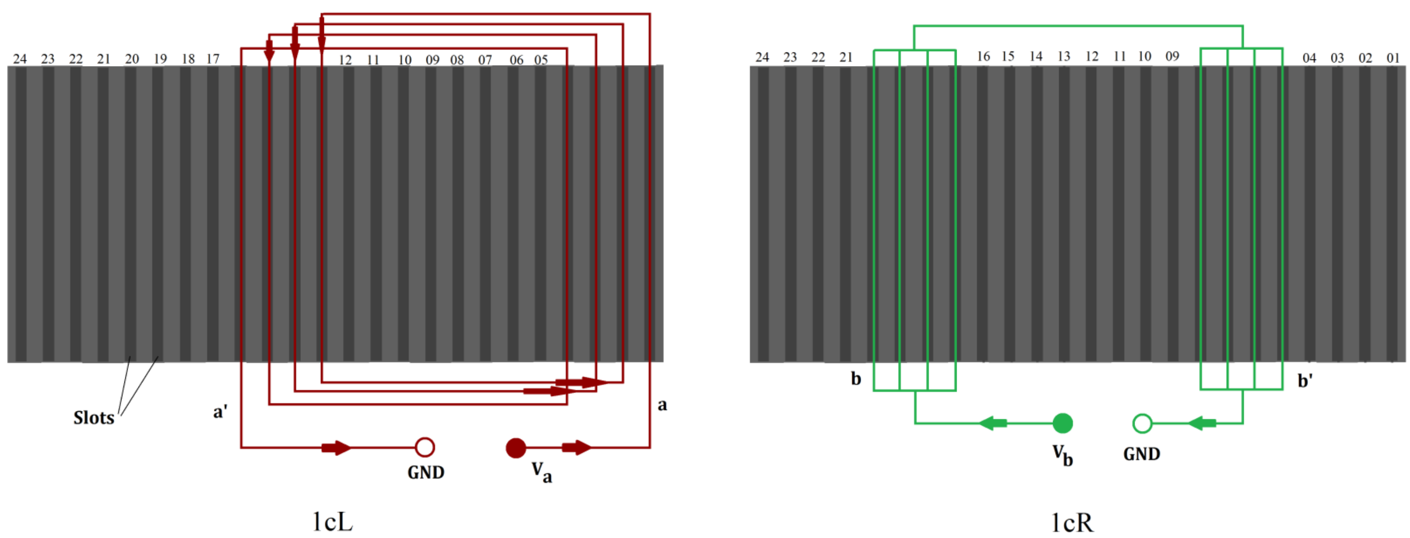


Figure 1c : *Plane projection views of the stator core with partial windings installed. The core is the same as in Fig. 1b. 1cL shows the phase* a *windings, which are in fact series wound. 1cR shows the phase* b *windings, which are parallel wound. In a practical motor however, the same topology is used for winding all the phases. The poor phase* c *has been left out of this schematic.*

The phases are typically labelled as *a*, *b* and *c*. It is natural to expect that each phase will be allotted an equal share of windings, and indeed this is what happens in practice. The layout of the windings is however non-trivial, and I can save a lot of words by showing in Fig. 1cL how the phase *a* windings fit into the stator (now opened out) of Fig. 1b. Note from this figure that the phase has two branches – the proper branch *a* and the primed branch *a'*. These are defined as per the convention that when the voltage applied in the concerned phase is positive, current flows upwards through the proper branch and downwards through the primed branch. In this figure, the slot numbers allotted to *a* are 01-04 and to *a'* are 13-16. Note that since the stator has 24 slots, diametrically opposite slots are separated by 12; hence the winding *a'* is in fact directly opposite to the winding *a*. By analogy it is clear that phase *b* should get slot numbers 05-08 and 17-20; in fact the correct way to go about it is to assign the first interval to *b'* and the second to *b*. This is shown in Fig. 1cR.

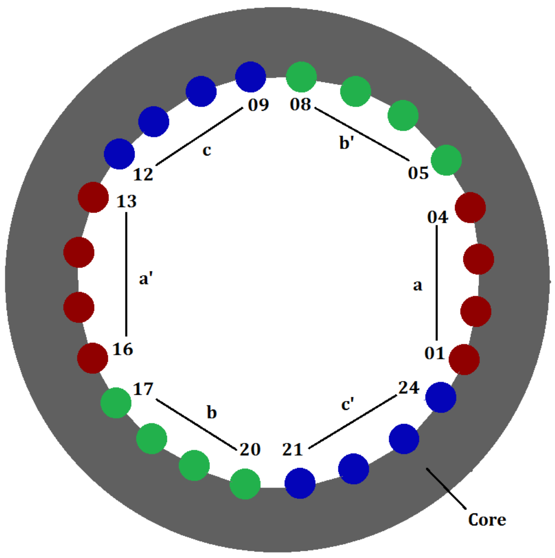
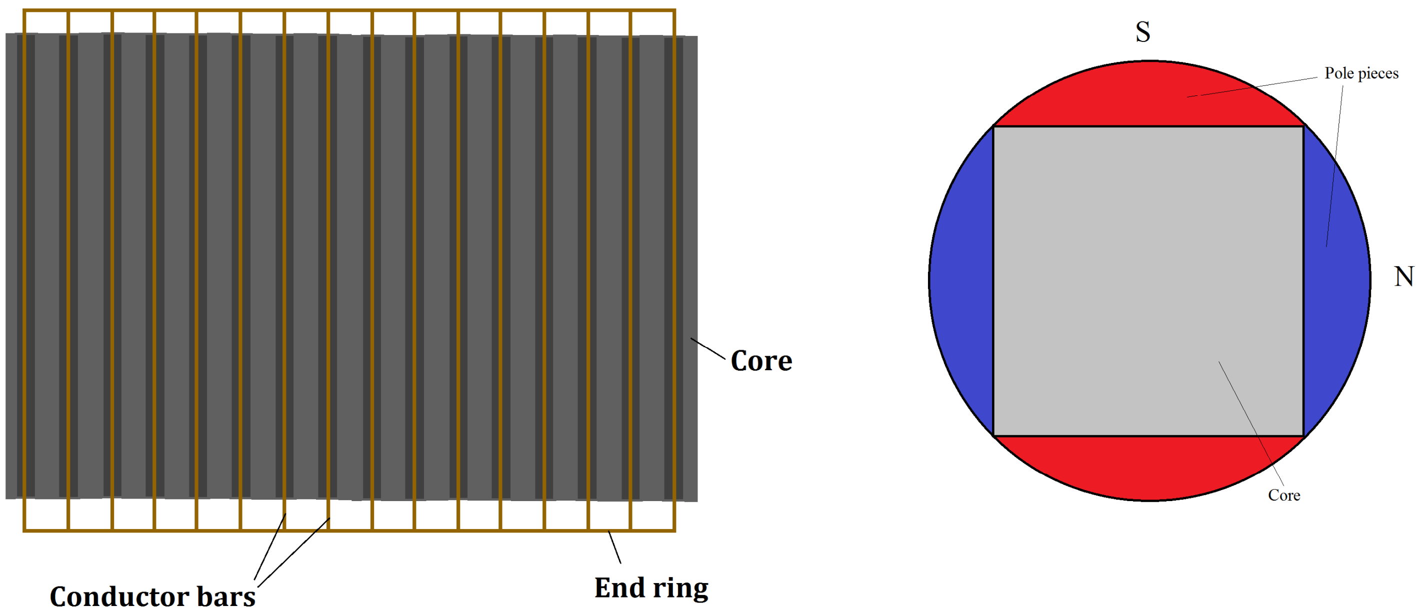


Figure 1d : *Top view of the stator core and all windings installed. The rotor fits into the space inside the stator.*



1eL

1eR

Figure 1e : *The rotors of IM and PMSM. 1eL shows the plane projection view of the rotor core with windings. These consist of conductor bars which are shorted by rings at the ends. 1eR shows top view of the PMSM rotor. Two pairs of North (blue) and South (red) poles are indicated hence it is a 4-pole rotor.*

Finally, the top view of the three-phase stator is shown in Fig. 1d. The windings are arranged in the order *a*-*b'*-*c*-*a'*-*b*-*c'* (while the ordering of the phases can be arbitrary, proper and primed branches must always alternate). Of course in a dc motor the stator has only one set of windings. They are arranged in a manner similar to any of the phases shown here. The structure of the rotor is simpler. In IM the rotor is just a set of conductor bars around the edge of the core short-circuited by rings on the top and bottom (Fig. 1eL – this structure is reminiscent of a squirrel cage). In PMSM the rotor is a permanent magnet and has a suitably shaped pole pair (Fig. 1eR). Finally, one more important motor parameter is its polarity. The stator of Fig. 1d is a 2-pole stator; in a 2*n*-pole motor there are *n* copies of the basic unit *a*-*b'*-*c*-*a'*-*b*-*c'* plastered round the stator surface. If the motor is PMSM then the rotor too has *n* pole pairs like 4-pole motor in Fig. 1eR.

This completes the basic description of the stator and rotor structure. We are now almost ready to start the analysis; before going for it there is just one other thing we need to look at and that is the motor drive. One possible way to run the motor is of course to connect the stator to a three-phase voltage supply. Indeed, that is just what is done in a lot of situations such as fans, pumps and crude grade lathes and cutters. This arrangement however has poor dynamic characteristics, which make it unsuitable for more demanding applications such as precision machines and railway locomotives. In such situations, the motor is connected to an inverter which separately regulates the voltage or current applied across each phase. The behaviour of the inverter is governed by an appropriate control algorithm; the inverter together with the controller is called the motor drive.

There are two main classes of inverters –

- Voltage source : These inverters regulate the voltage applied across each phase.
- Current source : These inverters regulate the current flowing through each phase.

Within each type there are again two main sub-categories –

- Discrete inverter : Can only produce a set of discrete voltage/current values. Among these the simplest are two-level inverters which can only turn each phase ON at a constant voltage/current or OFF. (Despite their simplicity, they are anything but worthy of ridicule – one of the best motor control algorithms [8] uses a two-level voltage source.)
- Continuous inverter : Can produce any value of voltage/current (upto a certain limit). A common example is one which produces three phase voltages at any arbitrary frequency (called a variable frequency drive or VFD).

So far as the control algorithm is concerned, the study of that is meaningless at this point since we do not yet have any idea of the dynamic behaviour of the motor. More details on motor drives may be found in the comprehensive text [1] by Krishnan. And while I am at it, I would like to mention that the substitution 'driver' for 'drive', though popular, is incorrect.

1.3 The Maxwellian bicylinder model

§1. Space phasors and the Park transformation. I am now in a position to start modelling the motor. Assuming that the rotor and stator are coaxial cylinders, I can expand any quantity defined on either of them (say the stator voltage or the rotor current) in a Fourier series [9]. Clearly, the fundamental harmonic of this series will be of the form $\cos n\theta$ and $\sin n\theta$ where $2n$ is the polarity. I will now assume that only this harmonic is present; since higher harmonics are detrimental to performance, stators and rotors are generally designed to minimize their role and this assumption causes only a small error. Thus, I can write any quantity F defined on either cylinder as $F = P\cos n\theta + Q\sin n\theta$; since the two basis functions are linearly independent, there is no question of further simplification. Here we see the first connection between this model and the circuit model – that model talks about two sets of windings on both stator and rotor and these two windings correspond to the cosine and sine components.

The next step is to cast the problem in complex notation [10]. I will express the above F in the form $\mathbf{F} = P + \mathrm{j}Q$ where j denotes the imaginary unit. The complex **F** is called a space phasor or space vector (usually abbreviated to plain 'vector') and the motor model in terms of these variables is called the space phasor model. Further, the real part of a phasor is referred to as the direct or 'd' component and the imaginary part is called the quadrature or 'q' component. It may be noted that a space vector of this form is by no means a vector quantity in actual space – for example the voltage which is eminently a scalar quantity in real space can assume a sinusoidal distribution on the stator and thus become a space phasor. To avoid confusion, in this paper all space phasors will be denoted in bold face while actual vector quantities (such as magnetic field) will be denoted by overhead arrows.

Now suppose I am given a three phase stator and that I am given the voltages V_a, V_b and V_c applied to the three phases – how will I write this in space phasor form ? To solve this, I first note that for a 2-pole motor ($n=1$) the space phasor representation of any distribution points from the minimum of the distribution to the maximum and has a magnitude proportional to the amplitude of the distribution. Thus, if a voltage V_a is applied on phase a on the stator of Fig. 1d and the other two phases are idle, then the voltage vector will be some constant times $V_a\exp(\mathrm{j}0)$.

Likewise if only phase c is active, then the vector will the same constant times $V_c\exp j(2\pi/3)$ while if only phase b is active it will be the constant times $V_b\exp j(4\pi/3)$. Invoking the superposition principle and using a Fourier expansion to calculate the constant, I get for the stator of Fig. 1d

$$\mathbf{V} = \frac{2}{\pi}\left(V_a + V_c\,\mathrm{e}^{\mathrm{j}2\pi/3} + V_b\,\mathrm{e}^{\mathrm{j}4\pi/3}\right) \quad . \tag{101}$$

This is called the Park transformation [11]. Note that the coefficient $2/\pi$ obtains from assuming that the phase windings span the entire stator; it will get modified if the geometrical layout of the windings is different. It also follows from the geometry that for a $2n$-pole motor, the $2\pi/3$ and $4\pi/3$ in the exponents will be replaced by $2\pi/3n$ and $4\pi/3n$. Using this transformation I can get the voltage vector corresponding to any combination of phase voltages. Two examples are illustrative. For a three-phase two level inverter, each of the three phase voltages can be either V_0 or zero. Assuming polarity 2 and substituting all eight possibilities into (101) we see that this inverter can produce a total of seven voltage vectors i.e.

$$\mathbf{V} = \frac{2V_0}{\pi}\exp \mathrm{j}\left(\frac{2\pi k}{6}\right); k = 0,1,2,3,4,5 \quad , \tag{102}$$

and the zero vector $\mathbf{V}=0$. The second example features three phase sinusoidal voltages

$$\begin{aligned} V_a &= V_0 \cos\Omega t \\ V_b &= V_0 \cos\left(\Omega t + 2\pi/3\right) \quad . \\ V_c &= V_0 \cos\left(\Omega t + 4\pi/3\right) \end{aligned} \tag{103}$$

Writing the cosine as a sum of imaginary exponentials, substituting (103) into (101) and simplifying, I have

$$V = \frac{3V_0}{\pi}\exp \mathrm{j}\left(\Omega t\right) \quad . \tag{104}$$

From the definition of space phasors, $\exp j(\Omega t)$ denotes $\cos(\theta-\Omega t)$ which is in fact a rotating wave. This shows that three phase voltages applied to the three phase stator create a rotating voltage vector, which enables the creation of a rotating magnetic field. This completes the discussion on space phasor notation and I now return to the problem of constructing the motor's dynamic model.

§2. Currents, vector potentials and magnetic fields. The quantities relevant for the problem are not hard to seek. There are the rotor and stator applied voltages and the currents flowing through them. Voltage is of course a scalar in real space; since the currents are constrained by the geometry to flow in the z direction (i.e. along the cylinder axis) they too become scalars. These currents give rise to magnetic fields $\vec{\mathrm{B}}$. These are all vectors in real space so I must deal with three components of each of them, and it appears that the calculation is going to become extremely complicated.

Two assumptions save the day at this point. The first is that the cylinder height is infinite compared to its radius. This immediately makes the problem two-dimensional with the z direction becoming redundant. The second assumption is that the rotor and stator conductors are thin compared to their radii. This means that the currents in these elements can be treated as surface currents i.e. currents per unit length. The magnetic fields will have a radial (ρ) and a tangential (θ) component. Now there are two physical steps in which $\vec{\mathrm{B}}$ plays a role – one is in inducing emfs in the rotor through motional effect and the other is in producing the torque on the rotor through the $i\vec{\mathrm{l}}\times\vec{\mathrm{B}}$ effect. It is easy to see that in both these cases, it is the radial component B_ρ which is relevant. Hence B too may be treated as a scalar in real space.

Now, calculating the vector potential and magnetic field for a sinusoidally distributed surface current on a cylinder is a standard exercise in electromagnetism [12]. The first step is to write $\vec{\mathrm{B}}=\nabla\times\vec{\mathrm{A}}$ where $\vec{\mathrm{A}}$ is the vector potential; I then take its curl. From Ampere's law, this equals $\mu_0\vec{\mathrm{J}}$ where $\vec{\mathrm{J}}$ is the current density vector; on the other side of the equation I can always make $\vec{\mathrm{A}}$ divergenceless by a suitable gauge transformation and thus reduce the double curl to a negative Laplacian. This is a Poisson equation; now since there is no current density in bulk space, it further reduces to a Laplace equation. This is solved using separation of variables and exploiting the fact that at any current-carrying surface, the tangential component of $\vec{\mathrm{B}}$ undergoes a jump discontinuity proportional to the surface current. In the subsequent discussion only the result and not the formalism behind this calculation will be used so I omit any further technical details and directly quote the answers. For a surface current $\vec{\mathrm{K}} = K_0 \cos n\theta \hat{\mathbf{z}}$ on the surface of a single cylinder of radius x, the vector potential is

$$A_z = \frac{\mu_0 K_0 \rho^n}{2nx^{n-1}} \cos n\theta \quad , \tag{105}$$

inside the cylinder and

$$A_z = \frac{\mu_0 K_0 x^{n+1}}{2n\rho^n} \cos n\theta \quad , \tag{106}$$

outside it. Taking the curl yields

$$B_\rho = \frac{\mu_0 K_0 \rho^{n-1}}{2x^{n-1}} (-\sin n\theta) \quad , \tag{107}$$

inside the cylinder and

$$B_\rho = \frac{\mu_0 K_0 x^{n+1}}{2\rho^{n+1}} (-\sin n\theta) \quad , \tag{108}$$

outside it. In space phasor terms these can be written as follows : for a surface current phasor $\mathbf{K}_0$, the vector potential phasor is

$$\mathbf{A}_{\text{inside}} = \frac{\mu_o \rho^n}{2nx^{n-1}} \mathbf{K}_0 \quad , \tag{109a}$$

$$\mathbf{A}_{\text{outside}} = \frac{\mu_0 x^{n+1}}{2n\rho^n} \mathbf{K}_0 \quad , \tag{109b}$$

and the magnetic field is

$$\mathbf{B}_{\text{inside}} = -\mathrm{j} \frac{\mu_0 \rho^{n-1}}{2x^{n-1}} \mathbf{K}_0 \quad , \tag{110a}$$

$$\mathbf{B}_{\text{outside}} = -\mathrm{j} \frac{\mu_0 x^{n+1}}{2\rho^{n+1}} \mathbf{K}_0 \quad . \tag{110b}$$

Note that the conversion from surface current $\mathbf{K}$ to current $\mathbf{i}$ is easy – the total current in the rotor or stator is just the surface current multiplied by the circumferential length of the windings. The relevant length is that of one branch of one phase winding for the stator and $1/2n$ of the circumference of a squirrel cage rotor. Because the surface currents are of fundamental importance in the modeling, I will keep these rather than the actual currents in the eventual dynamical equation. For convenience however I will refer to the surface currents as just 'currents' in the subsequent discussion.

At this point I am ready to introduce the MBC model. Let the rotor have a radius r and the stator have a radius R (since the rotor generally lies inside the stator I will take $r<R$). Let the conductivities of the wires/bars in the rotor and stator be σ and σ' respectively, and let the thicknesses of these conductors be b and b' (these symbols are also doing duty as phase names, but there should be no confusion). It is reasonable to assume that both rotor and stator cores are made of the same magnetic material, which is linear with permeability μ_c. When a current in any element produces a magnetic field in that element, this will be the relevant permeability. When a current in any element produces a field in the other element however, the effective permeability will be reduced because of the air gap between them. Let this modified permeability be μ_g where $\mu_g<\mu_c$. Finally, because the conductors along the circumference of the rotor and stator are separated by a gap, I define a conductor separation factor c and c' as the ratio of the angular span of one conductor to the total angular span between the centres of two adjacent conductors. Finally, I let the angular velocity of rotation of the rotor about its axis be ω in the anticlockwise (positive) direction.

Now suppose the rotor carries a current $\mathbf{K}_1$ which is a function of time. Since the typical size of a motor is of the order of 1m and the typical frequency is of order 100 Hz, the product gives a speed much smaller than that of light and the magnetostatic approximation is valid. Then the vector potential created by $\mathbf{K}_1$ at the rotor surface will be

$$\mathbf{A}_{r,1} = \frac{\mu_c r}{2n} \mathbf{K}_1 \quad , \tag{111}$$

while the potential at the stator surface will be

$$\mathbf{A}_{s,1}=\frac{\mu_g r^{n+1}}{2nR^n}\mathbf{K}_1=\frac{\mu_g}{\mu_c}\left(\frac{r}{R}\right)^{n+1}\frac{\mu_c R}{2n}\mathbf{K}_1 \quad . \tag{112}$$

In the above two equations, the subscripts r and s denote the rotor and stator respectively; the notation $\mathbf{A}_{s,1}$ means $\mathbf{A}$ at the stator surface on account of $\mathbf{K}_1$. The electric field induced in the stator follows as $\mathbf{E}$=-d$\mathbf{A}$/dt from Faraday's law; I multiply this field by the conductivity to get the two-dimensional current density and then multiply by the conductor thickness to get the surface current. Finally I scale this down by the conductor separation factor to account for the fact that the current is in fact zero along part of the circumference (where there is no conductor). Combining these, I get an induced stator current

$$\mathbf{K}_{s,1}=\frac{\mu_g}{\mu_c}\left(\frac{r}{R}\right)^{n+1}\frac{c'\mu_c\sigma' Rb'}{2n}\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)\mathbf{K}_1 \quad . \tag{113}$$

I can repeat this process for the rotor; there is however an extra term to be taken care of here. This arises because the rotor conductors are moving through a magnetic field, and an emf $\vec{\mathrm{E}}=\vec{\mathrm{v}}\times\vec{\mathrm{B}}$ is induced over and above the –d$\mathbf{A}$/dt effect. Viewing from the stator frame, the speed of motion of the rotor conductors is ωr which using (110) for $\mathbf{B}$ and converting electric field to surface current yields

$$\mathbf{K}_{r,1}=\frac{c\mu_c\sigma rb}{2n}\left(-\frac{\mathrm{d}}{\mathrm{d}t}+\mathrm{j}n\omega\right)\mathbf{K}_1 \quad . \tag{114}$$

Thus $\mathbf{K}_{r,1}$ is a correctional term on the original $\mathbf{K}_1$; this 'correctional' approach can in fact be used to eventually get the dynamic model [13] although I will follow a shorter procedure here. As an aside I would like to mention that if I view the dynamics from a frame rotating at arbitrary angular velocity ω_e with respect to the stator then the terms -d/dt and -d/dt+j$n\omega$ in (113) and (114) will get replaced by -d/dt-j$n\omega_e$ and -d/dt+j$n(\omega-\omega_e)$ respectively. This procedure, called arbitrary frames transformation, is useful in some cases though I will not employ it in this paper.

In a like manner I can calculate what will happen if the stator carries a current $\mathbf{K}_2$ which is a function of time. Repeating the steps I have

$$\mathbf{K}_{s,2}=\frac{c'\mu_c\sigma' Rb'}{2n}\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)\mathbf{K}_2 \quad , \tag{115}$$

and

$$\mathbf{K}_{r,2}=\frac{\mu_g}{\mu_c}\left(\frac{r}{R}\right)^{n-1}\frac{c\mu_c\sigma rb}{2n}\left(-\frac{\mathrm{d}}{\mathrm{d}t}+\mathrm{j}n\omega\right)\mathbf{K}_2 \quad . \tag{116}$$

It is readily seen that terms of the form $c\mu_c\sigma rb/2n$ are appearing repeatedly in these expressions. This quantity has the dimension of time and I make the variable definitions

$$\tau_r=\frac{c\mu_c\sigma rb}{2n} \quad , \tag{117a}$$

$$\tau_s=\frac{c'\mu_c\sigma' Rb'}{2n} \quad . \tag{117b}$$

In fact, as will be apparent from the subsequent development, τ_r and τ_s are the time constants of the rotor and stator. It is also convenient to make the definitions

$$\delta_1=\frac{\mu_g}{\mu_c}\left(\frac{r}{R}\right)^{n-1} \quad , \tag{118a}$$

$$\delta_2=\frac{\mu_g}{\mu_c}\left(\frac{r}{R}\right)^{n+1} \quad , \tag{118b}$$

and in terms of these quantities I can rewrite (113)-(116) as

$$\mathbf{K}_{r,1}=\tau_r\left(-\frac{\mathrm{d}}{\mathrm{d}t}+\mathrm{j}n\omega\right)\mathbf{K}_1 \quad , \tag{119a}$$

$$\mathbf{K}_{s,1} = \delta_2 \tau_s \left(-\frac{\mathrm{d}}{\mathrm{d}t} \right) \mathbf{K}_1 \quad , \tag{119b}$$

$$\mathbf{K}_{r,2} = \delta_1 \tau_r \left(-\frac{\mathrm{d}}{\mathrm{d}t} + \mathrm{j} n\omega \right) \mathbf{K}_2 \quad , \tag{119c}$$

$$\mathbf{K}_{s,2} = \tau_s \left(-\frac{\mathrm{d}}{\mathrm{d}t} \right) \mathbf{K}_2 \quad . \tag{119d}$$

With this I have assembled all the bricks and mortar needed to get the dynamic model. The only task left is actually building the house. While in my original works [13,14] I had done this floor by tedious floor, this time I will give an argument which will take me almost directly to the roof.

§3. The Voltage-current relationship. The first thing to get clear is what we are actually looking for. A dynamic model of an electrical device is generally a relation between the voltage applied on the device and the current developed in it. This is true for LCR circuits, devices like diodes and transistors and also more complex circuits like Chua circuits. Here too I will try to search for such a relation between the voltages $\mathbf{U}_r$ and $\mathbf{U}_s$ which I apply on the rotor and stator and the currents $\mathbf{K}_r$ and $\mathbf{K}_s$ which flow in them. Now in all these equations [(111) onwards] I have only talked about current and never voltage. But a voltage term is easy to see in the steps leading to (113) from (112); instead of multiplying the $\mathbf{E}$ by conductivity etc. I could have just multiplied it by the total length of the stator circuit to get the voltage $\mathbf{W}_{s,1}$ ('$\mathbf{U}$' is the applied voltage and '$\mathbf{V}$' has a more prominent role a little later) induced in the stator on account of $\mathbf{K}_1$. This voltage is of course proportional to $\mathbf{K}_{s,1}$ through some constant which I call Z'. If L' be the length of the stator circuit (note that L' is the total length of winding while l denotes the circumferential span), then $Z' = L'/\sigma' b'$. Analogously, the voltage across the rotor is proportional to the rotor current through $Z = L/\sigma b$ and I could have written (119a,b) in terms of $\mathbf{W}_{r,1}$ and $\mathbf{W}_{r,2}$ rather than $\mathbf{K}_{r,1}$ and $\mathbf{K}_{r,2}$. This Z term is just a dimensional constant which converts current to voltage; what we have to be careful about is what physically constitutes a current and what physically constitutes a voltage.

Suppose the net current in the rotor is given as $\mathbf{K}_r$ – I don't know how it has appeared but it has. I now want to find the voltage induced across the rotor and stator on its account. This of course follows from (119a,b) with an extra Z term :

$$\mathbf{W}_{r,r} = Z \tau_r \left(-\frac{\mathrm{d}}{\mathrm{d}t} + \mathrm{j} n\omega \right) \mathbf{K}_r \quad , \tag{120a}$$

$$\mathbf{W}_{s,r} = \delta_2 Z' \tau_s \left(-\frac{\mathrm{d}}{\mathrm{d}t} \right) \mathbf{K}_r \quad . \tag{120b}$$

Prima facie this may appear like a repeat of (119); while this is mathematically true, physically it has a vastly different interpretation. Equation (119) was merely describing the stator and rotor currents set up by some pre-existing $\mathbf{K}_1$ in the rotor while (120) is actually specifying the voltages induced across the rotor and stator on account of the total current $\mathbf{K}_r$ in the rotor. In a like manner I can write down the voltages induced across the rotor and stator on account of the net stator current $\mathbf{K}_s$ as

$$\mathbf{W}_{r,s} = \delta_1 Z \tau_r \left(-\frac{\mathrm{d}}{\mathrm{d}t} + \mathrm{j} n\omega \right) \mathbf{K}_s \quad , \tag{121a}$$

$$\mathbf{W}_{s,s} = Z' \tau_s \left(-\frac{\mathrm{d}}{\mathrm{d}t} \right) \mathbf{K}_s \quad . \tag{121b}$$

Now superpose; the total voltages $\mathbf{W}_r$ and $\mathbf{W}_s$ induced across rotor and stator are just the sum of the constituent parts i.e.

$$\mathbf{W}_r = \mathbf{W}_{r,r} + \mathbf{W}_{r,s} \quad , \tag{122a}$$

$$\mathbf{W}_s = \mathbf{W}_{s,r} + \mathbf{W}_{s,s} \quad . \tag{122b}$$

The last question to ask is : if I measure the voltages $\mathbf{X}_r$ and $\mathbf{X}_s$ across the rotor and stator with a voltmeter, what do I get ? The answer to this must be $\mathbf{X}_r = Z\mathbf{K}_r$ and $\mathbf{X}_s = Z'\mathbf{K}_s$. This follows from the fact that both the elements are linear conductors : if the current is $\mathbf{K}_r$, the electric field must be $\mathbf{K}_r/\sigma b$ and so the voltage must be $\mathbf{K}_r L/\sigma b$, which is $Z\mathbf{K}_r$. But these measured voltages $\mathbf{X}$ have to be the superposition of the applied voltages $\mathbf{U}$ and the induced voltages $\mathbf{W}$ – there are no other sources of voltage. Using this I get

$$\mathbf{X}_r - \mathbf{W}_r = \mathbf{U}_r \quad , \tag{123a}$$

$$\mathbf{X}_s - \mathbf{W}_s = \mathbf{U}_s \quad . \tag{123b}$$

Finally, I use the definitions of **X** and get **W** in terms of **K** by substituting (120) and (121) into (122); the result is the voltage-current dynamics of the motor :

$$\begin{bmatrix} Z\left[1+\tau_r\left(\frac{\mathrm{d}}{\mathrm{d}t}-\mathrm{j}n\omega\right)\right] & Z\delta_1\tau_r\left(\frac{\mathrm{d}}{\mathrm{d}t}-\mathrm{j}n\omega\right) \\ Z'\delta_2\tau_s\frac{\mathrm{d}}{\mathrm{d}t} & Z'\left(1+\tau_s\frac{\mathrm{d}}{\mathrm{d}t}\right) \end{bmatrix}\begin{bmatrix}\mathbf{K}_r \\ \mathbf{K}_s\end{bmatrix}=\begin{bmatrix}\mathbf{U}_r \\ \mathbf{U}_s\end{bmatrix} \quad . \tag{124}$$

This is the desired relationship. As a final step I take out Z and Z from the LHS and define the applied voltage vectors $\mathbf{V}_r = \mathbf{U}_r / Z$ and $\mathbf{V}_s = \mathbf{U}_s / Z'$ and in terms of these I get

$$\begin{bmatrix} 1+\tau_r\left(\frac{\mathrm{d}}{\mathrm{d}t}-\mathrm{j}n\omega\right) & \delta_1\tau_r\left(\frac{\mathrm{d}}{\mathrm{d}t}-\mathrm{j}n\omega\right) \\ \delta_2\tau_s\frac{\mathrm{d}}{\mathrm{d}t} & 1+\tau_s\frac{\mathrm{d}}{\mathrm{d}t} \end{bmatrix}\begin{bmatrix}\mathbf{K}_r \\ \mathbf{K}_s\end{bmatrix}=\begin{bmatrix}\mathbf{V}_r \\ \mathbf{V}_s\end{bmatrix} \quad . \tag{125}$$

§4. Newton's law. The modeling however is still not complete. The rotor angular velocity ω appears as a parameter in (125) but actually it is a dynamical variable (obviously, since motors accelerate and decelerate). So my next task is to find the equation satisfied by ω. Fortunately, this is much easier to obtain than the voltage-current dynamics. The torque of the motor follows from the good old $i\vec{\mathrm{I}}\times\vec{\mathrm{B}}$; since the torque is on the rotor, $\vec{\mathrm{I}}$ clearly has to mean the rotor current and $\vec{\mathrm{B}}$ has to mean the magnetic field at the rotor surface. This has contributions from both rotor and stator; however since an isolated spinning magnet cannot exert any torque on itself, the rotor field contribution to the torque must be zero. The contribution of the stator field at each point θ is proportional to $B_{r,s}(\theta)K_r(\theta)\mathrm{d}\theta$ and the total torque follows from integrating this over the entire cylinder. An identity in space phasor representations is

$$\int_0^{2\pi} F(\theta)G(\theta)\mathrm{d}\theta \propto \mathbf{F}\cdot\mathbf{G} \quad , \tag{126}$$

i.e. the integral of the product of two distributions is proportional to the dot product of their representative phasors, where the dot product is defined in the standard manner as the sum of the products of the respective direct and quadrature components. Using this and (110), the torque T of the motor emerges as

$$T = C\mathbf{K}_r \cdot (-\mathrm{j}\mathbf{K}_s) \quad , \tag{127}$$

where C is a positive constant. Its value can be found by carefully tracking [14] the coefficients giving forces and torques; without showing the details I just quote it as

$$C = \frac{\pi\mu_c\delta_1 r^2 h}{2} \quad . \tag{128}$$

This now enables me to write Newton's law for the motor : given the moment of inertia J of the motor and load and the drag torque Γ, I have

$$J\frac{\mathrm{d}\omega}{\mathrm{d}t} = C\mathbf{K}_r\cdot(-\mathrm{j}\mathbf{K}_s) - \Gamma \quad . \tag{129}$$

This completes the MBC model of the motor. It should be noted that these equations are in perfect agreement with the equivalent circuit dynamic models used by engineers. Equation (125) is fourth order (first order in each of two complex variables) while (129) is first order hence the resultant system is fifth order. It is highly nonlinear, on account of the product terms with ω in (125) and the current product in (129). As a final comment, I would like to mention that the form I have derived is independent of the type of motor. Nowhere in the actual modeling (§2 onwards) have I mentioned any specific details regarding the motor construction. I did introduce the Park transformation for a three phase stator but I have not used that result at any step of the main derivation. The only thing I have used is that functions defined on the cylinder can be expanded in harmonics, hence (125) and (129) are applicable for any cylindrical motor. In the next Subsection I will show how the equations for various motor types can be obtained by trivial substitution into the general structure.

1.3 IM and PMSM models

In this Subsection I will show the forms of (125) and (129) appropriate for IM and PMSM. The equation (129) will remain the same for all motors as it is perfectly general; (125) will depend on the type of motor and inverter. For IM, the rotor (Fig. 1eL) is just a short-circuited cylinder hence the voltage applied on it is zero. If the stator is connected to a voltage source inverter (VSI), the voltage-current equation reads

$$\begin{bmatrix} 1+\tau_r\left(\frac{\mathrm{d}}{\mathrm{d}t}-\mathrm{j}n\omega\right) & \delta_1\tau_r\left(\frac{\mathrm{d}}{\mathrm{d}t}-\mathrm{j}n\omega\right) \\ \delta_2\tau_s\frac{\mathrm{d}}{\mathrm{d}t} & 1+\tau_s\frac{\mathrm{d}}{\mathrm{d}t} \end{bmatrix}\begin{bmatrix}\mathbf{K}_r \\ \mathbf{K}_s\end{bmatrix}=\begin{bmatrix}0 \\ \mathbf{V}\end{bmatrix} , \tag{130}$$

in which the subscript 's' on V is unnecessary. On the other hand, if the stator is driven by a current source inverter (CSI) then $\mathbf{K}_s$ becomes a known function of time. Hence the second line of (130) goes out and the first line gives

$$\tau_r\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{K}_r+\left(1-\mathrm{j}\,\tau_r n\omega\right)\mathbf{K}_r=-\delta_1\tau_r\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{K}_s+\mathrm{j}\,\delta_1\tau_r n\omega\mathbf{K}_s \quad . \tag{131}$$

This procedure immediately tells us how go about the PMSM model. The rotor of that motor is a permanent magnet; it may be assumed as having an ingrained and constant current distribution $K_0\cos n\theta$ in its own frame, which translates to a current $K_0\cos n(\theta\text{-}\theta_r)$ in the stator frame where θ_r is the angle made by the rotor relative to stator-fixed axes. Clearly, $\mathrm{d}\theta_r/\mathrm{d}t=\omega$. Thus for the PMSM, $\mathbf{K}_r=K_0\exp\mathrm{j}\left(n\theta_r\right)$ where K_0 is a constant. The first line of (130) must go out, leaving behind

$$\tau_s\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{K}_s+\mathbf{K}_s=\mathbf{V}-\delta_2\tau_s K_0 n\omega\exp\mathrm{j}\left(n\theta_r\right) \quad . \tag{132}$$

These three equations are the common special cases of (125).

A useful situation which is studied is the steady state torque-speed (T vs. ω) characteristics [7] under applied three phase sinusoidal voltages at fixed frequency Ω. These characteristics follow from setting ω to be constant and trying out ansatzes (...)expj(Ωt) for the currents in the dynamical equations. I will not dwell on the details of the procedure but will indicate the results. For IM, the torque is very low at low values of ω and increases with ω upto a certain critical speed which is slightly less than Ω. Thereafter it plummets sharply as ω increases further and becomes zero at $\omega=\Omega$. The quantity $\Omega\text{-}\omega$ is called the slip frequency ε; at low ε the torque goes as ε while high ε it goes as $1/\varepsilon$. PMSM on the other hand generates positive torque only for $\omega=\Omega$. The relevant quantity is the phase difference φ between the stator and rotor current waveforms; the torque goes as $\sin\varphi$ and φ is called the torque angle.

In more sophisticated applications, the applied voltages/currents are not necessary sinusoidal but can have an arbitrary profile. The purpose of a motor control algorithm is in fact to select a proper function $\mathbf{V}(t)$ for a VSI or $\mathbf{K}_s(t)$ for a CSI which will result in smooth and high-performance operation of a motor. The control rule specifies the applied voltage/current as a function of time or of some of the motor parameters like induced stator voltage. Solving the motor equations with the control rule incorporated indicates the feasibility and quality of the strategy. One such solution has been proposed by Isao Takahashi and Toshihiko Noguchi [8] who are the inventors of direct torque control. Rather surprisingly this solution uses techniques which are quite different from the standard nonlinear dynamical tools. In the next Section I will propose two novel control algorithms and will use the nonlinear dynamic theory to prove their feasibility.

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